2943. Proposed by L. E. DICKSON, University of Chicago.

In the game of bridge, what is my chance: (*a*) that my hand will contain 4 aces? (*b*) that some hand will contain 4 aces? (*c*) that my hand will be a Yarborough, *i.e.*, contain no honor?

2944. Proposed by S. A. COREY, Des Moines, Iowa.

A particle of mass m, starting from rest, is drawn by a string over a smooth horizontal plane, the other end of the string moving in the plane with uniform acceleration n along a line perpendicular to the initial position of the string. Prove that the tension of the string is $3mn \cos \theta$, where θ is the angle which the string makes with the given line. Also prove that the motion of the particle is vibratory.

2945. Proposed by T. M. BLAKSLEE, Ames, Iowa.

A point P in the plane of the triangle ABC rotates in a given direction around the vertices taken in either cyclical order, in each case through an angle equal to the corresponding angle of the triangle. That is, for example, AP rotates around A through an angle equal to the angle A of the triangle; then BP around B through an angle equal to the angle B, and so on. Prove that P coincides with its original position at the end of six of these rotations. (See problem 2899, *1921*, 228.)

2946. Proposed by H. C. BRADLEY, Massachusetts Institute of Technology.

Cut a regular hexagon into the smallest number of pieces that can be fitted together to form an equilateral triangle: (*a*) no piece to be turned over; (*b*) some pieces may be turned over.

2947. Proposed by D. H. MENZEL, Princeton University.

An oil tank has the shape of a cylinder with ends which are segments of a sphere and with horizontal axis. The diameter of the cylinder being given, and the radius of the spherical segments, derive a formula that will express the volume of the liquid contained in the tank in terms of its depth.

2948. Proposed by J. B. REYNOLDS, Lehigh University.

Find the envelope of the normal planes to the curve,

$$x = a \cos t, \qquad y = a(1 - \cos t), \qquad \text{and} \qquad z = a \sin t.$$

2949. Proposed by J. B. REYNOLDS, Lehigh University.

Find the lateral area of the cone with vertex at $(0, 0, h)$ and whose base is the epicycloid,

$$x = \tfrac{3}{2}a \cos \theta - \tfrac{1}{2}a \cos 3\theta, \qquad y = \tfrac{3}{2}a \sin \theta - \tfrac{1}{2}a \sin 3\theta.$$

2950. Proposed by T. M. SIMPSON, JR., Randolph-Macon College, Ashland, Va.

Determine the curve which cuts the radius vector at an angle proportional to the radius vector.

NOTES.

25. The Area of a Quadrilateral.—The first expression for the area of an inscribed convex quadrilateral, in terms of its sides, was given by Brahmagupta (born 598 A.D.), without proof, in the following form:[1] "The product of half the sides and countersides is the gross area of a triangle and tetragone. Half the sum of the sides set down four times, and severally lessened by the sides, being multiplied together, the square root of the product is the exact area." The latter result appeared in a treatise written by Mahāvīrācārya, about 850 A.D., who gives[2] "The rule for arriving at the minutely accurate measurement of the

[1] *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bháscara.* Translated and edited by H. T. Colebrooke, London, 1817, pp. 295–296. See H. Weissenborn "Das Trapez bei Euklid" *Abhandlung zur Geschichte der Mathematik*, Heft 2, Suppl. historisch-literarischen Abtheilung, *Zeitschrift für Math. u. Physik*, vol. 4, 1879, pp. 181–184.

[2] *The Ganita-Sāra-Sangraha of Mahāvīrācārya* with English translations and notes. By M. Raṅgācārya, Madras, 1912, p. 198. Neither Brahmagupta nor Mahāvīrācārya knew that their rule for the exact area of a quadrilateral was only true for cyclic figures. The inexactness of the rule for all quadrilaterals was first pointed out by Bháskara (*l.c.*, § 167) who was born 1114 A.D.

area (of trilateral and quadrilateral figures):—. . . Four quantities represented (respectively) by half the sum of the sides as diminished by (each of) the sides (taken in order) are multiplied together; and the square root (of the product so obtained) gives the minutely accurate measure (of the area of the figure)." From this the area of a trilateral figure may be represented algebraically as[1] $\sqrt{[s(s-a)(s-b)(s-c)]}$, where s is half the sum of the sides whose lengths are a, b, and c; while the area of the quadrilateral figure[2] is

$$A = \sqrt{[(s-a)(s-b)(s-c)(s-d)]}, \tag{1}$$

where s is half the sum of the sides whose lengths are a, b, c, d. Moreover both Brahmagupta (*l.c.*, pp. 300–301) and Mahāvirācārya (*l.c.*, p. 199) state what is equivalent to the following expressions for the lengths of the diagonals of the quadrilateral:

$$m = \sqrt{\left[\frac{(ac+bd)(ab+cd)}{ad+bc}\right]}, \qquad n = \sqrt{\left[\frac{(ac+bd)(ad+bc)}{ab+cd}\right]}. \tag{2}$$

Formula (1) was rediscovered by Snellius who gives it in his commentary on the first book of Ludolphe van Ceulen's *De problematibus miscellaneis*.[3] The statement of the rule is given by Ozanam, without proof or reference to any other writer, in his *Cours de Mathematique* (Paris, vol. 3, 1690, p. 212; nouvelle édition, 1699, p. 147). Noticing this result Philip Naudé gave two demonstrations[4] of extraordinary complexity[5] to which Euler pays his respects introductory to furnishing a geometrical proof.[6]

Taking A, B, C, D as the vertices, in order, of an inscribed quadrilateral, and supposing AB, DC to meet in E, Euler first finds the following expression for the area, Q, of the quadrilateral,

$$QQ = \frac{1}{16}\cdot\frac{(AD-BC)(BE+CE+BC)}{BC}\cdot\frac{(AD-BC)(BE+CE-BC)}{BC}$$
$$\cdot\frac{(AD+BC)(BC+BE-CE)}{BC}\cdot\frac{(AD+BC)(BC-BE+CE)}{BC}.$$

He then shows that, apart from the numerical factor, the successive factors

[1] A result for which Heron of Alexandria gave an elegant proof hundreds of years earlier. The result was discovered by Archimedes who flourished still earlier.

[2] In connection with the following formulae of this note it will be supposed, unless otherwise stated, that a convex quadrilateral only is considered. The modifications for other cases are not difficult; some of them are later noted.

[3] *Oeuvres mathématiques de Ludolphe Van Ceulen*, traduites du hollandais en latin et enrichies de notes, par Snellius. Leyden, 1619. This was about 200 years before the first publication of Brahmagupta's results.—See Chasles, *Aperçu historique sur l'origine et le développement des méthodes en géométrie* . . . 2e édition, Paris, 1875, pp. 292, 432.

[4] "Demonstratio trium theorematum," *Miscellanea Berolinensia* . . . tome 3, 1727, pp. 259–269.

[5] C. L. A. Kunze comments on the proof which "zwar streng und bündig, aber mit einer *unerträglichen Weitschweifigkeit* abgefasst ist" (*Ueber einige theils bekannte, teils neue Sätze vom Dreieck und Viereck*, zweite vermehrte Ausgabe, Halle, 1848, p. 4).

[6] L. Euler, "Variae demonstrationes geometriae," *Nova acta acad. sc. Petrop.*, vol. 1 (1747–1748), 1750, pp. 57–63.

of the right-hand member are, respectively, $2S - 2BC$, $2S - 2DA$, $2S - 2AB$, $2S - 2CD$, where $2S = BC + DA + AB + CD$; whence he arrives at the form

$$Q = \sqrt{[(S - AB)(S - BC)(S - CD)(S - DA)]}.$$

A neat trigonometrical derivation of this result was given by Fuss[1] in a paper of considerable interest for the history of Poncelet polygons. Several other results here given will be noted in what follows.

The area was given in determinant form by Dostor:[2]

$$16A^2 = -\begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix}.$$

Such a form would be suggested naturally to anyone familiar with formula (1), and noticing the factors of this determinant, with a substituted for $-a$, given by Ferrers[3] in 1861.

In 1782 Lhuilier gave,[4] in effect, the following expression for the radius, R, of the circle circumscribing the quadrilateral:

$$R = \frac{1}{4}\left[\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}\right]^{\frac{1}{2}}. \tag{3}$$

For the numerator, within brackets, of the right-hand member, Klügel-Mollweide-Grunert substituted:[5] $abcd\Sigma a^2 + (abcd)^2\Sigma(1/a)^2$.

It was early recognized that, in general, there are three inscribed convex quadrilaterals with sides of given lengths; that these were equivalent in area but not superposable; that they have only three diagonals different in length. In 1626 Albert Girard stated[6] that the product of the lengths of the three different diagonals divided by twice the diameter of the circumcircle, is equal to the area of each of the quadrilaterals.

Hence, from (3), it appears that the product of the lengths of the three diagonals is $[(ab + cd)(ac + bd)(ad + bc)]^{1/2}$ which would be expected, since, from "Ptolemy's theorem," or (2), the product of the diagonals (m, n) of the quadrilateral whose sides are consecutively a, b, c, d, is given by $mn = ac + bd$.

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[1] N. Fuss, "De quadrilateris quibus circulum tam inscribere quam circumscribere licet," Conuentui exhib. die 14. Aug 1794, *Nova acta acad. sc. Petrop.*, vol. 10 (1792), 1797, pp. 103–125.

[2] G. Dostor, *Eléments de la théorie des déterminants*, Paris, 1877, p. 198.

[3] N. M. Ferrers, *An Elementary Treatise on Trilinear Coördinates*, Cambridge, 1861, p. 70.

[4] S. Lhuilier, *De Relatione Mutua . . . seu de Maximis et Minimis*, Warsaw, 1782, p. 21. Formula (3), in just this form, was given by Fuss (*l.c.*), p. 105.

[5] Klügel-Mollweide-Grunert, *Mathematisches Wörterbuch*, fünfter Theil, zweiter Band, Leipsic, 1831, p. 875.

[6] A. Girard, *Tables de sinus, tangentes et sécantes selon le raid de 10000 parties . . .*, 1626; Dutch translation, 1629. Compare Kästner, *Geschichte der Mathematik*, Göttingen, vol. 3, 1799, pp. 107–110; and S. Günther, *Vermischte Untersuchungen zur Geschichte der mathematischen Wissenschaften*, Leipsic, 1876, pp. 17–21.

The area, A', of the diagonal triangle of an inscribed quadrilateral was given by Dostor[1] in 1868 as

$$A' = \frac{4a^2b^2c^2d^2A}{(a^2b^2 - c^2d^2)(a^2d^2 - b^2c^2)};$$

whence Dostor found

$$\frac{A}{A'} = \frac{1}{4}\left(\frac{a^2}{c^2} - \frac{b^2}{d^2} + \frac{c^2}{a^2} - \frac{d^2}{b^2}\right) = \frac{1}{4}\left(\frac{ab}{cd} - \frac{cd}{ab}\right)\left(\frac{ad}{bc} - \frac{bc}{ad}\right).$$

With the above notation Dostor showed,[2] in 1848, that in the case of any quadrilateral,

$$16A^2 = 4m^2n^2 - (a^2 - b^2 + c^2 - d^2)^2. \tag{4}$$

He showed also that if the quadrilateral is circumscribed about a circle

$$\begin{aligned} A &= \tfrac{1}{2}\sqrt{[(mn + ac - bd)(mn - ac + bd)]}, \\ &= \tfrac{1}{2}\sqrt{[(mn + a'a'' - b'b'' + c'c'' - d'd'')(mn - a'a'' + b'b'' - c'c'' + d'd'')]}, \end{aligned} \tag{5}$$

where a', a''; b', b''; c', c''; d', d'' are the segments of the respective sides formed by the points of contact such that $a'' = b'$, $b'' = c'$, $c'' = d'$, $d'' = a'$.

Dostor found (*l.c.*, 1848, p. 73) that if p and q are the lengths of the line segments joining the middle points of pairs of opposite sides of any quadrilateral,

$$A = \tfrac{1}{2}\sqrt{[(p + q + m)(p + q - m)(p + q + n)(p + q - n)]}.$$

In 1874, he gave[3] yet another expression in terms of the coördinates of the four vertices:

$$2A = \begin{vmatrix} 1 & 0 & x_1 & y_1 \\ 0 & 1 & x_2 & y_2 \\ 1 & 0 & x_3 & y_3 \\ 0 & 1 & x_4 & y_4 \end{vmatrix}.$$

If a, b, c, d are the consecutive sides of a convex quadrilateral, and $\delta(\neq 0)$ is the length of the line joining the middle points of the principal diagonals, Catalan found[4] that

$$16\delta^2 A = (d^2 - b^2)\sqrt{[4a^2c^2 - (a^2 + c^2 - 4\delta^2)^2]} + (c^2 - a^2)\sqrt{[4b^2d^2 - (b^2 + d^2 - 4\delta^2)^2]}.$$

Strehlke showed[5] that if A and C are a pair of opposite angles of the quadrilateral

[1] G. Dostor, *Propriétés Nouvelles des Quadrilatères en Général* . . . Greifswald, [1868], p. 28; also in *Archiv der Mathematik und Physik*, vol. 48. In connection with this result E. W. Hobson erroneously omits the factor 4 (*A Treatise on Plane Trigonometry*, second edition, 1897, p. 205; fourth edition, 1918, p. 208).

[2] G. Dostor, *Nouvelles Annales de Mathématiques*, vol. 7, 1848, pp. 70 and 230; also vol. 33, 1874, p. 563.

[3] *Nouvelles Annales de Mathématiques*, vol. 33, 1874, p. 562; also *Archiv der Mathematik und Physik*, vol. 56, 1874, p. 240.

[4] E. Catalan, *Nouvelle Correspondance Mathématique*, vol. 6, 1880, pp. 52–53.

[5] Strehlke, "Zwei neue Sätze vom ebenen und sphärischen Viereck" . . ., *Archiv der Mathematik und Physik*, vol. 2, 1842, p. 324. J. F. König found (*Archiv* . . ., vol. 34, 1860, p. 14), for the area of a spherical quadrilateral the following expression which, in part, reminds one of Strehlke's formula:

$$A^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\tfrac{1}{2}(A+C). \tag{6}$$

Hence[1] from formula (4)

$$\left.\begin{aligned}16A^2 &= 4(ac+bd)^2 - (a^2-b^2+c^2-d^2)^2 - 16abcd\cos^2\tfrac{1}{2}(A+C),\\ &= 4(ac-bd)^2 - (a^2-b^2+c^2-d^2)^2 + 16abcd\sin^2\tfrac{1}{2}(A+C).\end{aligned}\right\} \tag{7}$$

If a circle can be inscribed in the quadrilateral, $a + c = b + d$, and this formula becomes[2]

$$A^2 = abcd\sin^2\tfrac{1}{2}(A+C).$$

If the quadrilateral is also inscribable[3] (Fuss, *l.c.*, p. 114),

$$A^2 = abcd; \tag{8}$$

under these conditions the radius of the inscribed circle is[4] $r = A/s = A/(a+c)$; for R, Klügel-Mollweide-Grunert wrote (*l.c.*)

$$R = \tfrac{1}{4}\sqrt{[\Sigma a^2 + abcd\Sigma(1/a)^2]}.$$

$$\sin\frac{E}{2} = \frac{\sqrt{[\sin(s-a)\sin(s-b)\sin(s-c)\sin(s-d) - \sin a\sin b\sin c\sin d\cos^2\frac{1}{2}(B+D)]}}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d}$$
$$+\frac{\sin a\sin b\sin B\cos\frac{1}{2}(c+d)\cos\frac{1}{2}(c-d) + \sin c\sin d\sin D\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{8\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d\cos^2\frac{1}{2}m},$$

where m is the diagonal AC, and E is the spherical excess of the quadrilateral $ABCD$. See also *Archiv. . . .*, vol. 4, 1843, p. 448.

[1] R. Baltzer, *Die Elemente der Mathematik*, Leipsic, vol. 2, 1862, p. 305.

[2] S. L. Loney, *Plane Trigonometry*, Cambridge, 1893, p. 253.

[3] Corresponding to formulæ (4), (5), and (8), P. Serret gave for the spherical quadrilateral, (*Des Méthodes en Géométrie*, Paris, 1855, p. 43):

$$\sin^2\frac{E}{2} = \frac{(\sin\frac{1}{2}m\sin\frac{1}{2}n + \cos\frac{1}{2}a\cos\frac{1}{2}c - \cos\frac{1}{2}b\cos\frac{1}{2}d)(\sin\frac{1}{2}m\sin\frac{1}{2}n + \cos\frac{1}{2}b\cos\frac{1}{2}d - \cos\frac{1}{2}a\cos\frac{1}{2}c)}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d}; \tag{4'}$$

$$\sin^2\frac{E}{2} = \frac{\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-c)\sin\frac{1}{2}(s-d)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d}; \tag{5'}$$

$$\sin^2\frac{E}{2} = \tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c\tan\tfrac{1}{2}d; \tag{8'}$$

where E is the spherical excess. From formula (5′) we get (Grunert, *Nouvelles Annales de Mathématiques*, vol. 22, 1863, p. 336)

$$\cos^2\frac{E}{2} = \frac{\cos\frac{1}{2}s\cos\frac{1}{2}(s-a-b)\cos\frac{1}{2}(s-a-c)\cos\frac{1}{2}(s-a-d)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d}.$$

[4] Fuss found this result (*l.c.*, p. 112) by considering the question: What shall be the radius of a circle to which a quadrilateral with given sides is circumscribed, in order that the area of the quadrilateral shall be a maximum? He showed that it was when the quadrilateral was cyclic, *i.e.*, inscribed also in a circle. He found also an expression for the distance between the centers of these circles in terms of their radii. In this connection one may consult "The in-and-circumscribed quadrilateral" by W. E. Byerly (*Annals of Mathematics*, vol. 10, pp. 123–128, 1909); it is there shown that the necessary and sufficient condition for two circles (radii r, R) the distance between whose centers is x, to have a quadrilateral circumscribed to the first and inscribed to the second is that

$$\frac{1}{r^2} = \frac{1}{(x+R)^2} + \frac{1}{(x-R)^2}.$$

Déprez gave the following relation between the diagonals and the radii: $mn/2r = r + \sqrt{r^2+4R^2}$ (*Mathesis*, vol. 8, 1888, pp. 80, 104, 237–239). See also O. Kolberg, Program Braunsberg, 1853–1856.

It is well known that when there is one quadrilateral at the same time circumscribed about one circle and inscribed in another, there are an infinite number of quadrilaterals so related to these circles. The question naturally arises: Which one of these quadrilaterals has the greatest area? Or the least area? Welsch showed (*L'Intermédiaire des Mathématiciens*, vol. 16, 1909, p. 37) that the former occurred when the diagonals of the quadrilateral were at right angles to one another; that is, when two opposite vertices of the quadrilateral were at the extremities of the diameter of the circumcircle, center O, passing through the center, I, of the in-circle. He showed also that the quadrilateral is a minimum when it is an isosceles trapezium whose parallel sides touch the in-circle where it is intersected by OI.

Dostor showed [1] that in every quadrilateral:

$$(a) \qquad A=\tfrac{1}{4}(a^2-b^2+c^2-d^2)\cdot|\tan(m,n)|,$$

when (m, n) is not $90°$; [2]

$$(b) \qquad A=\tfrac{1}{4}(m^2-n^2)\cdot|\tan(p,q)|,$$

when (p, q) is not $90°$; and

$$(c) \qquad A=\tfrac{1}{2}mn\sin(m,n).$$

If g and h, k and l, are the segments of the interior diagonals,

$$A=\tfrac{1}{2}(gl+lh+hk+kg)\sin(m,n),$$

a form which Kummer found useful in his discussion of a rational quadrilateral, that is, one whose sides, diagonals and area are rational numbers[3] (*Journal für die reine und angewandte Mathematik*, vol. 37, 1848, p. 11).

If the exterior angles of an inscribed quadrilateral are bisected, the area of the quadrilateral formed by these bisecting lines is (E. W. Hobson, *Treatise on Plane Trigonometry*, second edition, Cambridge, 1897, p. 222)

$$\frac{1}{2}\frac{s^2(ab+cd)(ad+bc)}{(a+c)(b+d)\sqrt{[(s-a)(s-b)(s-c)(s-d)]}}.$$

Meier Hirsch gave the following formula (*Sammlung geometrischer Aufgaben*, Erster Teil, Berlin, 1805, p. 36), for the area of a quadrilateral whose sides are known and in which a pair of opposite angles, (a, d) and (b, c) are equal:

$$A=\frac{1}{4}\frac{ad+bc}{ad-bc}\sqrt{[(a+b+c+d)(a+b-c-d)(a+d-b-c)(b+d-a-c)]}.$$

When the opposite angles are right angles P. F. Verhulst showed that we have

[1] G. Dostor, *Propriétés Nouvelles des Quadrilatères en Général* . . ., Greifswald, [1868], pp. 3, 6, 7.

[2] If the quadrilateral is circumscribable about a circle this formula becomes

$$A=\tfrac{1}{2}(bd-ac)\cdot\tan(m,n)$$

(C. Davison, *Subjects for Mathematical Essays*, London, 1915, p. 33).

[3] The problem of the rational quadrilateral has an extensive history; see the discussion by L. E. Dickson in this MONTHLY, *1921*, 244–250; and in his *History of the Theory of Numbers*, vol. 2, 1920, pp. 216–221.

(*Correspondance Mathématique et Physique* (Quetelet), Brussels, vol. 6, 1830, p. 121) $A = (s - a)(s - d) = (ad + bc)/2$.

Other expressions for the area are given in E. Heis and T. J. Eschweiler, *Lehrbuch der Geometrie, Dritter Teil, Ebene und sphärische Trigonometrie,* zweite Auflage, Köln, 1875, pp. 86–90, and in M. Hirsch, *l.c.*, pp. 33–41.

In 1782 Lhuilier considered the question of when the polygon with given sides should have a maximum area, and found that this occurred when it was inscribed in a circle.[1] The result for the quadrilateral follows at once on setting $A + C = 180°$, $\cos C = -\cos A$, in formula (6) or the first part of formula (7). Lhuilier considered also the question of when the area is a minimum and showed [2] that this arose when the quadrilateral was no longer convex but when the sides cut one another. From the second part of formula (7) it appears that the area is a minimum when $A + C = 0$, that is, when[3] $\cos C = \cos A$, the case of the inscribed concave quadrilateral. In this case

$$\begin{aligned} 16A_1^2 &= 4(ac - bd)^2 - (a^2 - b^2 + c^2 - d^2)^2, \\ &= (a+b+c+d)(-a-b+c+d)(-a+b-c+d)(-a+b+c-d), \\ &= -\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}, \end{aligned} \tag{9}$$

which is Ferrers's determinant referred to above. From this, it follows at once that for any convex quadrilateral

$$\sqrt{[s(s-a-d)(s-b-d)(s-c-d)]} \not> A,$$

a result given by Wolstenholme[4] in one of his problems. Among a number of other maxima and minima results connected with the quadrilateral, R. Sturm proved [5] that for given sides, in a given order, the inscribed convex quadrilateral has the greatest diagonal product and hence the greatest area (compare (4)).

In *Astronomische Nachrichten*, no. 42, November, 1823, Shumacher reports that on page 61 of the description by Möbius of the observatory at Leipzig the following problem occurs: Let $ABCDE$ be any five points in the plane joined so as to form the five triangles ABC, BCD, CDE, DEA, EAB whose areas α, β, γ, δ, ϵ, respectively, are known; required the area of the pentagon $ABCDE$.

[1] S. Lhuilier, *l.c.*, p. 5; see also Enneper "Ueber das Maximum eines Vierecks von gegebenen Seiten," *Gött. Nachrichten*, 1885, pp. 175–180. The same result holds true for a spherical quadrilateral (W. J. McClelland and T. Preston, *Treatise on Spherical Trigonometry*, part 2, London, 1903, p. 50.)

[2] S. Lhuilier, *l.c.*, pp. 23–24; formula (9) was given, in effect, by Lhuilier. Details of this question are discussed by R. Sturm and E. Lampe in *Journal für die reine und angewandte Mathematik*, vol. 96, 1884, pp. 78–80.

[3] R. Baltzer, *Die Elemente der Mathematik*, vol. 1, 1860, pp. 129–130; vol. 2, 1862, p. 306. For inscribed quadrilaterals $A^2 - A_1^2 = abcd$ (Möbius, *Journal für die reine und angewandte Mathematik*, vol. 3, 1828, pp. 17–18).

[4] J. Wolstenholme, *A Book of Mathematical Problems*, London, 1867, problem 377; third edition, 1878, problem 596. It is clear that Wolstenholme might have written "$< A$" instead of "$\not> A$."

[5] R. Sturm, *Maxima and Minima in der elementaren Geometrie*, Leipsic, 1910, p. 25.

Shumacher appends a solution handed to him by Gauss (*Carl Friedrich Gauss Werke*, vol. 4, 2ter Abdruck, 1880, pp. 406–407); this leads to the equation

$$A^2 - (\alpha + \beta + \gamma + \delta + \epsilon)A + (\alpha\beta + \beta\gamma + \gamma\delta + \delta\epsilon + \epsilon\alpha) = 0.$$

P. Serret noted (*Nouvelles Annales de Mathématiques*, vol. 7, 1848, p. 28) that if the pentagon becomes the quadrilateral $ABCD$, α, β, γ being the areas of the triangles ABC, BCD, CDA, the above equation for the area of the quadrilateral becomes $A^2 - (\alpha + \beta + \gamma)A + \alpha\beta + \beta\gamma = 0$,—which can, on solving, be at once verified.

The problem of the construction with ruler and compasses of an inscribed quadrilateral, being given its sides, has an interesting history extending over five hundred years. This is, in general, only a particular case of the problem discussed by Ozanam (*Dictionaire Mathematique*, Amsterdam, 1691, pages 461–464): "Construct the quadrilateral, of given area, given the lengths of its four sides." R. C. ARCHIBALD.

SOLUTIONS

2832 [1920, 227]. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that the square of the sum of four squares is the sum of four squares, that the square of the sum of eight squares is the sum of six squares, and that the square of the sum of sixteen squares is the sum of ten squares.

I. SOLUTION BY THE PROPOSER.

Given the identity,

$$\begin{aligned}
&(Pp+Pq+Qp-Qq-Rr+Rs-Sr-Ss-Tt-Tu+Ut-Uu-Vv-Vx+Xv-Xx)^2\\
&\quad+(Pr+Ps-Qr+Qs-Rp+Rq+Sp+Sq-Tv+Tx+Uv+Ux+Vt-Vu-Xt-Xu)^2\\
&\quad+(Pp-Pq-Qp-Qq+Rr+Rs-Sr+Ss+Tt-Tu+Ut+Uu+Vv-Vx+Xv+Xx)^2\\
&\quad+(Pr-Ps+Qr+Qs+Rp+Rq+Sp-Sq+Tv+Tx+Uv-Ux-Vt-Vu-Xt+Xu)^2\\
&\quad+(Pt+Pu+Qt-Qu-Rv-Rx+Sv-Sx+Tp+Tq-Up+Uq+Vr-Vs+Xr+Xs)^2\\
&\quad+(Pv-Px-Qv-Qx-Rt+Ru-St-Su+Tr+Ts-Ur+Us-Vp+Vq-Xp-Xq)^2\\
&\quad+(Pt-Pu-Qt-Qu+Rv-Rx+Sv+Sx-Tp+Tq-Up-Uq-Vr-Vs+Xr-Xs)^2\\
&\quad+(Pv+Px+Qv-Qx+Rt+Ru-St+Su-Tr+Ts-Ur-Us+Vp+Vq-Xp+Xq)^2\\
&\qquad\equiv 2(P^2+Q^2+R^2+S^2+T^2+U^2+V^2+X^2)(p^2+q^2+r^2+s^2+t^2+u^2+v^2+x^2). \qquad (1)
\end{aligned}$$

Add $(P^2 + Q^2 + R^2 + S^2 + T^2 + U^2 + V^2 + X^2)^2$ and $(p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + x^2)^2$ to each member of (1). The second member then becomes the square of the sum of 16 squares, and the first member becomes the sum of 10 squares, as required. If $P = Q = R = S = t = u = v = x =$ zero, it follows that the square of the sum of eight squares is the sum of six squares. That the square of the sum of four squares is the sum of four squares is a direct consequence of Euler's well known theorem: (sum of four squares) (sum of four squares) = (sum of four squares).

II. Mr. NORMAN ANNING, of the University of Michigan, contributes formulæ exhibiting these squares as the sum of 3, 5 and 9 squares, instead of 4, 6, and 10. The results are given for record.